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CONVERGENCE OF APPROXIMATED SEQUENCES FOR NONEXPANSIVE MAPPINGS

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1. INTRODUCTION

Let C be a closed, convex subset of a Hilbert space and let x be an element of C. Let T be a nonexpansive mapping from C into itself such that the set $F(T)$ of fixed points of T is nonempty. In 1967, Browder [3] showed the following convergence theorem for a nonexpansive mapping: *For each t with $0<t<1$, let x_t be an element of C satisfying*

$$x_t = tx + (1-t)Tx_t.$$

Then $\{x_t\}$ converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ as $t \downarrow 0$. This result was extended to a Banach space by Reich [12] and Takahashi and Ueda [23]. On the other hand, in the framework of a Hilbert space, Wittmann [24] studied the convergence of the iterated sequence which is defined by

$$y_0 = x, \quad y_{n+1} = a_n x + (1-a_n)Ty_n, \quad n = 0, 1, 2, \dots ,$$

where $\{a_n\}$ is a real sequence satisfying $0 \le a_n \le 1$ and $a_n \to 0$. Recently, using an idea of Browder [3], Shimizu and Takahashi [15] studied the convergence of the following approximated sequence for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

$$x_n = a_n x + (1-a_n)\frac{1}{n}\sum_{j=1}^{n} T^j x_n, \quad n = 1, 2, \dots ,$$

where $\{a_n\}$ is a real sequence satisfying $0 < a_n < 1$ and $a_n \to 0$. Shimizu and Takahashi [16] also studied the convergence of another iteration process for a family of nonexpansive mappings in the framework of a Hilbert space. The iteration process is a mixed iteration process of Wittmann's and Shimizu and Takahashi's. For simplicity, we state their iteration process in the case of a simple mapping:

$$y_0 = x, \quad y_{n+1} = a_n x + (1-a_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n, \qquad n = 0, 1, 2, \dots ,$$

where $\{a_n\}$ is a real sequence satisfying $0 \le a_n \le 1$ and $a_n \to 0$.

In this paper, we first extend Wittmann's result to a Banach space [17], which gives an answer to Reich's problem [13]. To extend his result, we essentially need the concept of a sunny, nonexpansive retraction [4, 11]. We also extend Shimizu and Takahashi's results to a Banach space [18, 19]. Then we show strong convergence theorems for an asymptotically nonexpansive semigroup [20] by the use of an asymptotically invariant sequence of means, which have been developed in the study of nonlinear ergodic theorems [1, 5, 6, 9, 10, 14, 21, 22].

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2. Preliminaries and notations

Throughout this paper, all vector spaces are real and we denote by $\mathbb{N}$ and $\mathbb{N}_+$, the set of all nonnegative integers and the set of all positive integers, respectively. We also denote $\max\{a,0\}$ by $(a)_+$ for a real number a.

Let E be a Banach space with norm $\|\cdot\|$. Let C be a subset of E and let T be a mapping from C into itself. We denote by $F(T)$, the set of fixed points of T. T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for each } x, y \in C.$$

T is said to be asymptotically nonexpansive with Lipschitz constants $\{k_n : n \in \mathbb{N}\}$ if $\overline{\lim}_n k_n \leq 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for each } x, y \in C \text{ and } n \in \mathbb{N}.$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ such that T is asymptotically nonexpansive with Lipschitz constants $\{k_n\}$.

Let $U = \{x \in E : \|x\| = 1\}$. E is said to be uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$ for each $x, y \in U$ with $\|x - y\| \geq \varepsilon$. We know [7] that if C is a closed, convex subset of a uniformly convex Banach space and T is an asymptotically nonexpansive mapping from C into itself such that $F(T)$ is nonempty then $F(T)$ is convex. Let E^* be the dual of E. The value of $y \in E^*$ at $x \in E$ will be denoted by $\langle x, y \rangle$. We also denote by J, the duality mapping from E into 2^{E^*}, i.e.,

$$Jx = \{y \in E^* : \langle x, y \rangle = \|x\|^2 = \|y\|^2\}, \qquad x \in E.$$

E is said to be smooth if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) exists uniformly for $x \in U$. We know that if E is smooth then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable then the duality mapping is norm to weak star, uniformly continuous on each bounded subset of E.

Let C be a convex subset of E, let K be a nonempty subset of C and let P be a retraction from C onto K, i.e., $Px = x$ for each $x \in K$. A retraction P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If there exists a sunny retraction from C onto K which is also nonexpansive, then K is said to be a sunny, nonexpansive retract of C. Concerning sunny, nonexpansive retractions, we know the following [4, 11]:

Proposition 1. *Let E be a smooth Banach space and let C be a convex subset of E. Let K be a nonempty subset of C and let P be a retraction from C onto K. Then P is sunny and nonexpansive if and only if*

$$\langle x - Px, J(y - Px) \rangle \leq 0 \qquad \text{for each } x \in C \text{ and } y \in K.$$

Hence there is at most one sunny, nonexpansive retraction from C onto K.

In the case when E is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, C is a closed, convex subset of E and K is a closed, convex subset of C, there is a mapping P from C onto K which satisfies

$$(2.2) \qquad \|x - Px\| = \min_{y\in K} \|x - y\| \qquad \text{for each } x \in C.$$

This mapping P is said to be a metric projection from C onto K. We know that a metric projection is nonexpansive and that a mapping P from C onto K satisfies (2.2) if and only if $\langle x - Px, y - Px\rangle \le 0$ for each $y \in K$ and $x \in C$. So in this case, the metric projection is the unique sunny, nonexpansive retraction.

Let S be a semigroup and let $B(S)$ be the space of all bounded real valued functions defined on S with supremum norm. For each $s \in S$ and $f \in B(S)$, we define elements l_sf and r_sf in $B(S)$ by

$$(l_sf)(t) = f(st) \quad \text{and} \quad (r_sf)(t) = f(ts), \qquad t \in S.$$

Let X be a subspace of $B(S)$ containing 1 and let X^* be its dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. Let X be l_s-invariant for each $s \in S$, i.e., $l_s(X) \subset X$. A mean μ on X is said to be left invariant if $\mu(l_sf) = \mu(f)$ for each $s \in S$ and $f \in X$. A sequence $\{\mu_n\}$ of means on X is said to be strongly left regular if

$$\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0 \quad \text{for each } s \in S,$$

where l_s^* is the adjoint operator of l_s. Let X be l_s and r_s-invariant for each $s \in S$, i.e., $l_s(X) \subset X$ and $r_s(X) \subset X$. A mean μ on X is said to be invariant if $\mu(l_sf) = \mu(r_sf) = \mu(f)$ for each $s \in S$ and $f \in X$. A net $\{\mu_\alpha\}$ of means on X is said to be asymptotically invariant if

$$\lim_\alpha \big(\mu_\alpha(l_sf) - \mu_\alpha(f)\big) = 0 \quad \text{and} \quad \lim_\alpha \big(\mu_\alpha(r_sf) - \mu_\alpha(f)\big) = 0 \quad \text{for each } s \in S \text{ and } f \in X.$$

Let H be a Hilbert space and let C be a closed, convex subset of H. A family $\mathcal{S} = \{T_t : t \in S\}$ of mappings is said to be a uniformly Lipschitzian semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ if

(i) k_t is a nonnegative real number for each $t \in S$ and $\sup_{t\in S} k_t < \infty$;
(ii) for each $t \in S$, T_t is a mapping from C into itself and $\|T_tx - T_ty\| \le k_t\|x - y\|$ for each $x, y \in C$;
(iii) $T_{ts}x = T_tT_sx$ for each $t, s \in S$ and $x \in C$;

We denote by $F(\mathcal{S})$, the set of common fixed points of $\mathcal{S}$, i.e., $\cap_{s\in S}\{x \in C : T_tx = x\}$. A uniformly Lipschitzian semigroup $\mathcal{S} = \{T_t : t \in S\}$ on C with Lipschitz constants $\{k_t : t \in S\}$ is said to be asymptotically nonexpansive if $\inf_{s\in S}\sup_{t\in S} k_{st} \le 1$, and it is said to be nonexpansive if $k_t = 1$ for all $t \in S$. If S is left reversible, i.e., each two right ideals of S have nonempty intersection, S is naturally directed by $t \le s$ if and only if $\{t\} \cup tS \supset \{s\} \cup sS$ for $t, s \in S$. So, in this case, $\inf_s \sup_t k_{st} = \overline{\lim}_t k_t$. Let $\mathcal{S} = \{T_t : t \in S\}$ be a uniformly Lipschitzian semigroup on C such that $\{T_tx : t \in S\}$ is bounded for some $x \in C$ and let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \|T_tx - y\|^2$ is an element of X for each $x \in C$ and $y \in H$. For each mean μ on X and $x \in C$, there is a unique element x_0 of C satisfying

$$\mu_t\langle T_tx, y\rangle = \langle x_0, y\rangle \qquad \text{for all } y \in H,$$

where $\mu_t\langle T_tx, y\rangle$ is the value of μ at the function $t \mapsto \langle T_tx, y\rangle$. According to [14], we write such x_0 by $T_\mu x$. We remark that $T_\mu x = x$ for $x \in F(\mathcal{S})$.

3. Convergence theorems for a mapping

The following celebrated convergence theorem of an approximated sequence for a nonexpansive mapping was established by Browder [3]:

Theorem 1 (Browder). *Let C be a closed, convex subset of a Hilbert space, let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty and let P be the metric projection from C onto $F(T)$. Let x be an element of C and for each t with $0 < t < 1$, let x_t be a unique point of C which satisfies*

$$x_t = tx + (1-t)Tx_t. \tag{3.1}$$

Then $\{x_t\}$ converges strongly to Px as t tends to 0.

This theorem was extended to a Banach space by Reich [12] and Takahashi and Ueda [23]. From their results, their proofs and Proposition 1, we know the following:

Theorem 2 (Reich, Takahashi and Ueda). *Let C be a closed, convex subset of a Banach space whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Let x be an element of C and let x_t be a unique element of C which satisfies (3.1) for each t with $0 < t < 1$. Assume that each nonempty, T-invariant, bounded, closed, convex subset of C contains a fixed point of T. Then $\{x_t\}$ converges strongly to an element of $F(T)$. Moreover, for each element x of C, define $Px = \lim_t x_t$. Then P is a sunny, nonexpansive retraction from C onto $F(T)$.*

Theorem 1 and Theorem 2 induced Halpern [8] and Reich [13] to study the convergence of the iteration

$$y_0 = x, \quad y_{n+1} = a_n x + (1-a_n)Ty_n, \quad n \in \mathbb{N}, \tag{3.2}$$

where $\{a_n\}$ is a real sequence such that $0 \le a_n \le 1$ and $a_n \to 0$. They obtained partial results and posed problems for the convergence of the sequence defined by (3.2). Since Halpern studied the problem in the framework of a Hilbert space, we introduce Reich's problem [13]:

Problem 1 (Reich). *Let E be a Banach space. Is there a sequence $\{a_n\}$ such that whenever a weakly compact, convex subset C of E possessed the fixed point property for nonexpansive mappings, then the sequence $\{y_n\}$ defined by (3.2) converges to a fixed point of T for all x in C and all nonexpansive $T : C \to C$?*

Recently, Wittmann [24] solved the problem in the case when E is a Hilbert space:

Theorem 3 (Wittmann). *Let C be a closed, convex subset of a Hilbert space, let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty and let P be the metric projection from C onto $F(T)$. Let x be an element of C and let $\{a_n\}$ be a real sequence which satisfies*

$$0 \le a_n \le 1, \ \lim_{n\to\infty} a_n = 0, \ \sum_{n=0}^{\infty} a_n = \infty \ \text{ and } \ \sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty. \tag{3.3}$$

Then the sequence $\{y_n\}$ defined by (3.2) converges strongly to Px.

We extend Wittmann's result to a Banach space [17]. The difficulty to prove it depends on that the duality mapping is not weakly continuous in a Banach space. In a Hilbert space, the duality mapping is essentially the identity mapping and hence it is weakly continuous.

Theorem 4. *Let C be a closed, convex subset of a Banach space whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Let $\{a_n\}$ be a real sequence which satisfies* (3.3). *Let x be an element of C and let $\{y_n\}$ be the sequence defined by* (3.2). *Assume that $\{x_t\}$ converges strongly to $z \in F(T)$ as $t \downarrow 0$, where for each t with $0 < t < 1$, x_t is a unique point of C which satisfies* (3.1). *Then $\{y_n\}$ converges strongly to z.*

So we solve Reich's problem as follows from Theorem 2 and Theorem 4:

Theorem 5. *Let C, T, $\{a_n\}$, x and $\{y_n\}$ be as in Theorem* 4. *Assume that each nonempty, T-invariant, bounded, closed, convex subset of C contains a fixed point of T. Let P be the sunny, nonexpansive retraction from C onto $F(T)$. Then $\{y_n\}$ converges strongly to Px.*

On the other hand, Shimizu and Takahashi [15] studied the convergence of another approximated sequence for an asymptotically nonexpansive mapping in the framework of a Hilbert space:

Theorem 6 (Shimizu and Takahashi). *Let C be a closed, convex subset of a Hilbert space, let T be an asymptotically nonexpansive mapping from C into itself with Lipschitz constants $\{k_n\}$ such that $F(T)$ is nonempty and let P be the metric projection from C onto $F(T)$. Let $0 < a < 1$, let $b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1 - k_j| + e^{-j})$ and let $a_n = \frac{b_n - 1}{b_n - 1 + a}$ for $n \in \mathbb{N}_+$. Let x be an element of C and let x_n be a unique point of C which satisfies*

$$x_n = a_n x + (1 - a_n)\frac{1}{n}\sum_{j=1}^{n} T^j x_n, \qquad n \in \mathbb{N}_+.$$

Then $\{x_n\}$ converges strongly to Px.

We extend the result to a Banach space. First, we show that $F(T)$ is a sunny, nonexpansive retract for an asymptotically nonexpansive mapping T in a Banach space [18]:

Theorem 7. *Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be an asymptotically nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Then $F(T)$ is a sunny, nonexpansive retract of C.*

Now we show a generalization of Shimizu and Takahashi's result [18]:

Theorem 8. *Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be an asymptotically nonexpansive mapping from C into itself with Lipschitz constants $\{k_n\}$ such that $F(T)$ is nonempty and let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{a_n\}$ be a real sequence such that*

$$0 < a_n \le 1, \quad \lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{b_n - 1}{a_n} < 1,$$

where $b_n = \sum_{j=0}^{n} k_j/(n+1)$ for $n \in \mathbb{N}$. Let x be an element of C and for all sufficiently large n, let x_n be a unique point of C which satisfies

$$x_n = a_n x + (1 - a_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n. \tag{3.4}$$

Then $\{x_n\}$ converges strongly to Px.

Remark 1. The inequality $\overline{\lim}_n (b_n - 1)/a_n < 1$ yields $(1-a_n)b_n < 1$ for all sufficiently large n. So for such n, there exists a unique point x_n of C satisfying $x_n = a_n x + (1-a_n)\frac{1}{n+1}\sum_{j=0}^n T^j x_n$, since the mapping T_n from C into itself defined by $T_n u = a_n x + (1-a_n)\frac{1}{n+1}\sum_{j=0}^n T^j u$ satisfies $\|T_n u - T_n v\| \le (1-a_n)b_n\|u-v\|$ for all $u, v \in C$.

In the case when T is nonexpansive, we have the following [18]:

Theorem 9. *Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty and let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \le 1$ and $\lim_n a_n = 0$. Let x be an element of C and for each $n \in \mathbb{N}$, let x_n be a unique point of C which satisfies* (3.4). *Then $\{x_n\}$ converges strongly to Px.*

Recently, Shimizu and Takahashi [16] studied the convergence of another iteration process for a family of nonexpansive mappings. The iteration process is a mixed iteration process of (3.2) and (3.4). For simplicity, we state their result for a nonexpansive mapping:

Theorem 10 (Shimizu and Takahashi). *Let C be a closed, convex subset of a Hilbert space, let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty and let P be the metric projection from C onto $F(T)$. Let $\{a_n\}$ be a real sequence which satisfies*

$$0 \le a_n \le 1, \quad \lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} a_n = \infty. \tag{3.5}$$

Let x be an element of C and let $\{y_n\}$ be the sequence defined by

$$y_0 = x, \quad y_{n+1} = a_n x + (1-a_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n, \qquad n \in \mathbb{N}. \tag{3.6}$$

Then $\{y_n\}$ converges strongly to Px.

We also extend their result to a Banach space [19]. From Theorem 7, we know that $F(T)$ is a sunny, nonexpansive retract for an asymptotically nonexpansive mapping T.

Theorem 11. *Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be an asymptotically nonexpansive mapping from C into itself with Lipschitz constants $\{k_n\}$ such that $F(T)$ is nonempty. Let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{a_n\}$ be a real sequence which satisfies* (3.5) *and*

$$\sum_{n=0}^{\infty}\left((1-a_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j\right)^2 - 1\right)_+ < \infty.$$

Let x be an element of C and let $\{y_n\}$ be the sequence defined by (3.6). *Then $\{y_n\}$ converges strongly to Px.*

In the case when T is nonexpansive, we have the following [19]:

Theorem 12. *Let C be a closed, convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Let P be the sunny, nonexpansive retraction from C onto $F(T)$. Let $\{a_n\}$ be a real sequence which satisfies* (3.5). *Let x be an element of C and let $\{y_n\}$ be the sequence defined by* (3.6) *Then $\{y_n\}$ converges strongly to Px.*

4. Convergence theorems for families of mappings

In 1975, Baillon [1] proved the first nonlinear ergodic theorem in the framework of a Hilbert space:

Theorem 13 (Baillon). *Let C be a closed, convex subset of a Hilbert space and let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Then for each $x \in C$, the Cesàro means*

$$\frac{1}{n+1}\sum_{i=0}^{n} T^i x$$

converges weakly to an element of $F(T)$.

Using an asymptotically invariant net of means, Rodé [14] and Takahashi [21] generalized Baillon's theorem. From their results, we know the following:

Theorem 14 (Rodé, Takahashi). *Let C be a closed, convex subset of a Hilbert space and let S be a semigroup such that there exists an invariant mean on $B(S)$. Let $\mathcal{S} = \{T_t : s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then there exists a nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $T_tP = PT_t = P$ for each $t \in S$ and $Px \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in C$. Moreover, let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on $B(S)$. Then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to Px.*

We show that Theorem 13 is a direct consequence of Theorem 14: *Let C and T be as in Theorem* 13 *and let x be an element of C. For each $n \in \mathbb{N}$, let μ_n be the mean on $B(\mathbb{N})$ defined by $\mu_n(f) = \frac{1}{n+1}\sum_{i=0}^{n} f_i$ for $f = (f_0, f_1, \cdots) \in B(\mathbb{N})$. It is easy to see that $\{T^n : n \in \mathbb{N}\}$ is a nonexpansive semigroup, $F(\{T^n : n \in \mathbb{N}\}) = F(T)$, $\{\mu_n\}$ is asymptotically invariant and $T_{\mu_n} x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x$ for each $n \in \mathbb{N}$. From Theorem* 14, *there exists a mapping P from C onto $F(T)$ and $\frac{1}{n+1}\sum_{i=0}^{n} T^i x$ converges weakly to Px.* So Theorem 14 is a generalization of Theorem 13. Moreover, many theorems can be reduced from Theorem 14; see [9, 10].

Let C and S be as in Theorem 14, let $\mathcal{S} = \{T_t : s \in S\}$ be an asymptotically nonexpansive semigroup on C, let x be an element of C, let P be the metric projection from C onto $F(\mathcal{S})$ and let $\{\mu_n\}$ be a sequence of means on $B(S)$. By the results in Section 3 and Theorem 14, it is natural to consider the following problems:

Problem 2. *Let $\{a_n\}$ be a real sequence such that $0 < a_n \le 1$ and $a_n \to 0$. Then does the sequence $\{x_n\}$ defined by*

$$x_n = a_n x + (1 - a_n) T_{\mu_n} x_n, \quad n \in \mathbb{N}$$

converge strongly to Px under some conditions?

Problem 3. *Let $\{b_n\}$ be a real sequence such that $0 \le b_n \le 1$ and $b_n \to 0$. Then does the sequence $\{y_n\}$ defined by*

$$y_0 = x, \quad y_{n+1} = b_n x + (1 - b_n) T_{\mu_n} y_n, \quad n \in \mathbb{N}$$

converge strongly to Px under some conditions?

In this section, we give answers to the problems in the framework of a Hilbert space. The first result in this section gives an answer to Problem 2 [20]. It is a generalization of Theorem 6 for an asymptotically nonexpansive semigroup:

Theorem 15. *Let C be a closed, convex subset of a Hilbert space H and let S be a semigroup. Let $\mathcal{S} = \{T_t : t \in S\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k_t : t \in S\}$ such that $F(\mathcal{S})$ is nonempty and let P be the metric projection from C onto $F(\mathcal{S})$. Let X be a subspace of $B(S)$ such that $1 \in X$, X is l_s-invariant for each $s \in S$, the mapping $t \mapsto \|T_t u - v\|^2$ is an element of X for each $u \in C$ and $v \in H$ and the mapping $t \mapsto k_t$ is an element of X. Let $\{\mu_n : n \in \mathbb{N}\}$ be a strongly left regular sequence of means on X. Let $\{a_n\}$ be a real sequence satisfying*

$$0 < a_n \le 1, \quad \lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{(\mu_n)_t(k_t) - 1}{a_n} < 1.$$

Let x be an element of C and let $\{x_n\}$ be the sequence defined by

$$x_n = a_n x + (1 - a_n) T_{\mu_n} x_n \tag{4.1}$$

for $n \ge n_0$, where n_0 is some natural number. Then $\{x_n\}$ converges strongly to Px.

Remark 2. By the similar reason as in Remark 1, there exists $n_0 \in \mathbb{N}$ such that there is a unique point $x_n \in C$ satisfying $x_n = a_n x + (1 - a_n) T_{\mu_n} x_n$ for $n \ge n_0$.

In the case when $\mathcal{S}$ is nonexpansive, we have the following [20]:

Theorem 16. *Let C, H, S, $\mathcal{S}$, P, X and $\{\mu_n\}$ be as in Theorem 15. Assume that $\mathcal{S}$ is nonexpansive, i.e., $k_t = 1$ for all $t \in S$. Let $\{a_n\}$ be a real sequence satisfying $0 < a_n \le 1$ and $\lim_n a_n = 0$. Let x be an element of C and let $\{x_n\}$ be the sequence defined by (4.1) for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to Px.*

Next, we give an answer to Problem 3 [20]. It is a generalization of Theorem 10 for an asymptotically nonexpansive semigroup:

Theorem 17. *Let C, H, S, $\mathcal{S}$, P, X and $\{\mu_n\}$ be as in Theorem 15. Let $\{b_n\}$ be a real sequence satisfying*

$$0 \le b_n \le 1, \quad \lim_{n\to\infty} b_n = 0, \quad \sum_{n=0}^{\infty} b_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \Big((1 - b_n)((\mu_n)_t(k_t))^2 - 1\Big)_+ < \infty.$$

Let x be an element of C and let $\{y_n\}$ be the sequence defined by

$$y_0 = x, \quad y_{n+1} = b_n x + (1 - b_n) T_{\mu_n} y_n, \quad n \in \mathbb{N}. \tag{4.2}$$

Then $\{y_n\}$ converges strongly to Px.

In the case when $\mathcal{S}$ is nonexpansive, we also have the following [20]:

Theorem 18. *Let C, H, S, $\mathcal{S}$, P, X and $\{\mu_n\}$ be as in Theorem 15. Assume that $\mathcal{S}$ is nonexpansive, i.e., $k_t = 1$ for all $t \in S$. Let $\{b_n\}$ be a real sequence satisfying $0 \leq b_n \leq 1$, $\lim_n b_n = 0$ and $\sum_{n=0}^{\infty} b_n = \infty$. Let x be an element of C and let $\{y_n\}$ be the sequence defined by (4.2). Then $\{y_n\}$ converges strongly to Px.*

5. Deduced theorems from the results in section 4

Throughout this section, we assume that C is a closed, convex subset of a Hilbert space H. Since we use abstract means in the results in Section 4, we can deduce many theorems from them. We give the proofs for some results in this section. For others, see [20]; see also [10]. First we extend Shimizu and Takahashi's results [15, 16].

Theorem 19. *Let T and U be asymptotically nonexpansive mappings from C into itself with Lipschitz constants $\{k_n : n \in \mathbb{N}\}$ and $\{\kappa_n : n \in \mathbb{N}\}$, respectively such that $TU = UT$ and $F(T) \cap F(U) \neq \emptyset$ and let P be the metric projection from C onto $F(T) \cap F(U)$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \leq 1$, $a_n \to 0$ and $\overline{\lim}_{n\to\infty}\Big(2\sum_{l=0}^{n}\sum_{i+j=l} k_i\kappa_j/(n+1)(n+2) - 1\Big)\Big/a_n < 1$ and let $\{b_n\}$ be a real sequence such that $0 \leq b_n \leq 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty}\Big((1-b_n)(2\sum_{l=0}^{n}\sum_{i+j=l} k_i\kappa_j/(n+1)(n+2))^2 - 1\Big)_+ < \infty$. Let x be an element of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n)\frac{2}{(n+1)(n+2)}\sum_{l=0}^{n}\sum_{i+j=l} T^i U^j x_n \qquad \textit{for all sufficiently large } n,$$

and

$$y_0 = x, \quad y_{n+1} = b_n x + (1-b_n)\frac{2}{(n+1)(n+2)}\sum_{l=0}^{n}\sum_{i+j=l} T^i U^j y_n \qquad \textit{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

Theorem 20. *Let T be an asymptotically nonexpansive mapping from C into itself with Lipschitz constants $\{k_n : n \in \mathbb{N}\}$ such that $F(T)$ is nonempty and let P be the metric projection from C onto $F(T)$. Let $\{r_n\}$ be a real sequence such that $0 < r_n < 1$ and $\lim_n r_n = 1$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \leq 1$, $a_n \to 0$ and $\overline{\lim}_n\Big((1-r_n)\sum_{i=0}^{\infty} r_n^i k_i - 1\Big)\Big/a_n < 1$ and let $\{b_n\}$ be a real sequence such that $0 \leq b_n \leq 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty}\Big((1-b_n)((1-r_n)\sum_{i=0}^{\infty} r_n^i k_i)^2 - 1\Big)_+ < \infty$. Let x be an element of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n)(1-r_n)\sum_{i=0}^{\infty} r_n^i T^i x_n \qquad \textit{for all sufficiently large } n,$$

and

$$y_0 = x, \quad y_{n+1} = b_n x + (1-b_n)(1-r_n)\sum_{i=0}^{\infty} r_n^i T^i y_n \qquad \textit{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

Proof. For each $n \in \mathbb{N}$, define a mean μ_n on $B(\mathbb{N})$ by $\mu_n(f) = (1-r_n)\sum_{i=0}^{\infty} r_n^i f_i$ for $f = (f_0, f_1, \cdots) \in B(\mathbb{N})$. Then for each $j \in \mathbb{N}$,

$$\begin{aligned}
\lim_{n\to\infty} \|\mu_n - l_j^*\mu_n\| &= \lim_{n\to\infty} \sup\left\{ \left| (1-r_n)\sum_{i=0}^{\infty} r_n^i f_i - (1-r_n)\sum_{i=0}^{\infty} r_n^i f_{i+j} \right| : f \in l^\infty, |f_i| \le 1 \right\} \\
&\le \lim_{n\to\infty} \sup\left\{ \left| (1-r_n)\sum_{i=0}^{j-1} r_n^i f_i \right| + \left| (1-r_n)\sum_{i=0}^{\infty} (r_n^{i+j} - r_n^i) f_{i+j} \right| : f \in l^\infty, |f_i| \le 1 \right\} \\
&\le \lim_{n\to\infty} 2(1-r_n^j) = 0.
\end{aligned}$$

So $\{\mu_n\}$ is strongly left regular. It is easy to see that $\{T^n : n \in \mathbb{N}\}$ is an asymptotically nonexpansive semigroup with Lipschitz constants $\{k_n\}$, $F(\{T^n : n \in \mathbb{N}\}) = F(T)$ and $T_{\mu_n}x = (1-r_n)\sum_{i=0}^{n} r_n^i T^i x$ for $n \in \mathbb{N}$. Hence by Theorem 15 and Theorem 17, we obtain the conclusion. □

The following is a generalization of Theorem 6 and Theorem 10; see also [2]. For simplicity, we state it for a nonexpansive mapping.

Theorem 21. *Let T be a nonexpansive mapping from C into itself such that $F(T)$ is nonempty and let P be the metric projection from C onto $F(T)$. Let $\{\alpha_{n,m} : n, m \in \mathbb{N}\}$ be a sequence of nonnegative real numbers such that $\sum_{m=0}^{\infty} \alpha_{n,m} = 1$ and $\lim_{n\to\infty} \sum_{m=0}^{\infty} |\alpha_{n,m+1} - \alpha_{n,m}| = 0$. Let $\{a_n\}$ be a real sequence such that $0 < a_n \le 1$ and $a_n \to 0$ and let $\{b_n\}$ be a real sequence such that $0 \le b_n \le 1$, $b_n \to 0$ and $\sum_{n=0}^{\infty} b_n = \infty$. Let x be an element of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n)\sum_{m=0}^{\infty} \alpha_{n,m} T^m x_n \qquad \textit{for } n \in \mathbb{N},$$

and

$$y_0 = x, \quad y_{n+1} = b_n x + (1-b_n)\sum_{m=0}^{\infty} \alpha_{n,m} T^m y_n \qquad \textit{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to Px.

We show some more results which can be deduced from the results in Section 4.

Theorem 22. *Let $\mathcal{S} = \{S(t) : t \in [0,\infty)\}$ be an asymptotically nonexpansive semigroup on C with Lipschitz constants $\{k(t) : t \in [0,\infty)\}$ such that $F(\mathcal{S})$ is nonempty, the mapping $t \mapsto k(t)$ is measurable and the mapping $t \mapsto \|S(t)u - v\|^2$ is measurable for each $u \in C$ and $v \in H$ and let P be the metric projection from C onto $F(\mathcal{S})$. Let $\{\gamma_n\}$ be a sequence of positive real numbers with $\gamma_n \to \infty$, let $\{a_n\}$ be a real sequence such that $0 < a_n \le 1$, $a_n \to 0$ and $\overline{\lim}_{n\to\infty}\left(\int_0^{\gamma_n} k(t)\,dt/\gamma_n - 1\right)\big/a_n < 1$ and let $\{b_n\}$ be a real sequence such that $0 \le b_n \le 1$, $b_n \to 0$, $\sum_{n=0}^{\infty} b_n = \infty$ and $\sum_{n=0}^{\infty}\left((1-b_n)(\int_0^{\gamma_n} k(t)\,dt/\gamma_n)^2 - 1\right)_+ < \infty$. Let x be an element of C and let $\{x_n\}$ and $\{y_n\}$ be the sequences defined by*

$$x_n = a_n x + (1-a_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} S(t)x_n\,dt \qquad \textit{for all sufficiently large } n,$$

and

$$y_0 = x, \quad y_{n+1} = b_n x + (1-b_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} S(t)y_n\,dt \qquad \textit{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* Px.

Proof. Let X be the space of all bounded measurable functions from $[0,\infty)$ into itself. We remark that an element f in X is not an equivalence class with the usual equivalence relation, where the usual equivalence relation $g \sim h$ means the Lebesgue measure of the set $\{t \in [0,\infty) : g(t) \neq h(t)\}$ is zero. The reason is that we consider that X is a subspace of $B([0,\infty))$ with the supremum norm. For each $n \in \mathbb{N}$, define a mean μ_n on $B(X)$ by $\mu_n(f) = \frac{1}{\gamma_n}\int_0^{\gamma_n} f(t)\,dt$ for $f \in B(X)$. It is easy to see that $\{\mu_n\}$ is strongly left regular and $S(\mu_n)x = \frac{1}{\gamma_n}\int_0^{\gamma_n} S(t)x\,dt$ for $n \in \mathbb{N}$. Hence by Theorem 15 and Theorem 17, we obtain the conclusion. □

Theorem 23. *Let* $\mathcal{S} = \{S(t) : t \in [0,\infty)\}$ *and* P *be as in Theorem 22. Let* $\{\lambda_n\}$ *be a sequence of positive real numbers with* $\lambda_n \to 0$, *let* $\{a_n\}$ *be a real sequence such that* $0 < a_n \leq 1$, $a_n \to 0$ *and* $\overline{\lim}_{n\to\infty}\Big(\lambda_n\int_0^\infty e^{-\lambda_n t}k(t)\,dt - 1\Big)\Big/a_n < 1$ *and let* $\{b_n\}$ *be a real sequence such that* $0 \leq b_n \leq 1$, $b_n \to 0$, $\sum_{n=0}^\infty b_n = \infty$ *and* $\sum_{n=0}^\infty\Big((1-b_n)(\lambda_n\int_0^\infty e^{-\lambda_n t}k(t)\,dt)^2 - 1\Big)_+ < \infty$. *Let* x *be an element of* C *and let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by*

$$x_n = a_n x + (1-a_n)\lambda_n\int_0^\infty e^{-\lambda_n t}S(t)x_n\,dt \qquad \text{for all sufficiently large } n,$$

and

$$y_0 = x, \quad y_{n+1} = b_n x + (1-b_n)\lambda_n\int_0^\infty e^{-\lambda_n t}S(t)y_n\,dt \qquad \text{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* Px.

Proof. Let X be as in the proof of Theorem 22. For each $n \in \mathbb{N}$, define a mean μ_n on $B(X)$ by $\mu_n(f) = \lambda_n\int_0^\infty e^{-\lambda_n t}f(t)\,dt$ for $f \in B(X)$. It is easy to see that $\{\mu_n\}$ is strongly left regular and $S(\mu_n)x = \lambda_n\int_0^\infty e^{-\lambda_n t}S(t)x\,dt$ for $n \in \mathbb{N}$. Hence by Theorem 15 and Theorem 17, we obtain the conclusion. □

The following is a generalization of the two theorems above. For simplicity, we state it for a nonexpansive semigroup.

Theorem 24. *Let* $\mathcal{S} = \{S(t) : t \in [0,\infty)\}$ *be a nonexpansive semigroup on* C *such that* $F(\mathcal{S})$ *is nonempty and the mapping* $t \mapsto \|S(t)u - v\|^2$ *is measurable for each* $u \in C$ *and* $v \in H$ *and let* P *be the metric projection from* C *onto* $F(\mathcal{S})$. *Let* $\{\alpha_n\}$ *be a sequence of measurable functions from* $[0,\infty)$ *into itself such that* $\int_0^\infty \alpha_n(t)\,dt = 1$ *for each* $n \in \mathbb{N}$, $\lim_{n\to\infty}\alpha_n(t) = 0$ *for almost every* $t \geq 0$, $\lim_{n\to\infty}\int_0^\infty |\alpha_n(t+s) - \alpha_n(t)|\,dt = 0$ *for all* $s \geq 0$ *and there exists* $\beta \in L^1_{\mathrm{loc}}[0,\infty)$ *such that* $\sup_n \alpha_n(t) \leq \beta(t)$ *for almost every* $t \geq 0$, *where* $\beta \in L^1_{\mathrm{loc}}[0,\infty)$ *means a restriction of* β *on* $[0,s]$ *belongs to* $L^1[0,s]$ *for each* $s > 0$. *Let* $\{a_n\}$ *be a real sequence such that* $0 < a_n \leq 1$ *and* $a_n \to 0$ *and let* $\{b_n\}$ *be a real sequence such that* $0 \leq b_n \leq 1$, $b_n \to 0$ *and* $\sum_{n=0}^\infty b_n = \infty$. *Let* x *be an element of* C *and let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences defined by*

$$x_n = a_n x + (1-a_n)\int_0^\infty \alpha_n(t)S(t)x_n\,dt \qquad \text{for } n \in \mathbb{N},$$

and

$$y_0 = x, \quad y_{n+1} = b_n x + (1-b_n)\int_0^\infty \alpha_n(t)S(t)y_n\,dt \qquad \text{for } n \in \mathbb{N},$$

respectively. Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* Px.

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